BLOCKING POLYHEDRA

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34

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PREFACE

In this Memorandum we investigate a duality relation-
ship (called "blocking") between nonnegative convex polyhedra.
The theory developed is then applied to a number of problems
in extremal combinatorics.

This Memorandum continues RAND's basic mathematical
work with network flows, graphs, and matroids. Earlier
relevant publications include R-375-PR, RM-5368-PR, and
RM-5375-PR.

SUMMARY

Let $\mathcal{B} = \{x \in R^n | Ax \geq 1, x \geq 0\}$, where A is a non-negative matrix and $1 = (1, \ldots, 1)$. The blocking poly-hedron of the polyhedron $\mathcal{B}$ is defined to be $\hat{\mathcal{B}} = \{x \in R^n | x \cdot \mathcal{B} \geq 1, x \geq 0\}$. It is shown that $\hat{\hat{\mathcal{B}}} = \mathcal{B}$ and a method is described for obtaining the minimal nonnegative matrix $B = \hat{A}$ (the blocker of A) such that $\hat{\mathcal{B}} = \{x \in R^n | Bx \geq 1, x \geq 0\}$.

The "max-flow min-cut equality" and the "length-width inequality" are always valid for a blocking pair of poly-hedra, and, in a sense, characterize the blocking relation.

Operations of "contraction" and "deletion," analogous to those in matroid theory, are defined for $\mathcal{B}$, and it is observed that, just as for matroids, a contraction in $\mathcal{B}$ corresponds to a deletion in $\hat{\mathcal{B}}$.

The geometric theory of blocking polyhedra is applied to various combinatorial situations in which A is taken to be the incidence matrix of a family of subsets of a finite set. A typical result in this domain is the following: The n by n permutation matrices, which are well known to be the extreme points of the n by n doubly stochastic matrices, are also the extreme points of the (unbounded) convex poly-hedron in n^2-space defined by the inequalities

$$\sum_{i \in I, j \in J} \xi_{ij} \geq |I| + |J| - n, \quad \text{all } I, J \subseteq \{1, \ldots, n\}$$

and

$$\xi_{ij} \geq 0, \quad \text{all } i, j \in \{1, \ldots, n\}.$$

BLOCKING POLYHEDRA

1. INTRODUCTION

It is well known that the permutation matrices can be
characterized geometrically as the extreme points of the
convex polyhedron of all doubly stochastic matrices. A
new result of this paper (Sec. 6) is that the permutation
matrices can also be characterized geometrically as the
extreme points of the following (unbounded) convex poly-
hedron. Let ξ_{ij} be a variable associated with cell i, j
of an n by n array, and consider the linear inequalities:

$$(1.1) \qquad \sum_{\substack{i \in I \\ j \in J}} \xi_{ij} \geq |I| + |J| - n, \quad \text{all } I, J \subseteq \{1, \ldots, n\},$$

$$(1.2) \qquad \xi_{ij} \geq 0, \quad \text{all } i, j \in \{1, \ldots, n\}.$$

Each extreme point of the polyhedron $\mathcal{P}$ defined in R^{n^2} by
(1.1) and (1.2) is a permutation matrix $x = (\xi_{ij})$. More-
over, almost all of the inequalities (1.1) that correspond
to positive right hand sides are essential in defining $\mathcal{P}$.
Thus we have another, albeit more complicated, geometric
representation of the permutation matrices.

The results of this paper were arrived at partly in
an attempt to understand better the inequalities (1.1).
The basic underlying geometric fact (Theorem 2.1) appears
to be a certain polarity between members of the class of

all (unbounded) convex polyhedra defined by linear in-
equalities of the form

(1.3) $Ax \geq 1$,

(1.4) $x \geq 0$,

where A is a nonnegative matrix, $0 = (0, \ldots, 0)$, and
$1 = (1, \ldots, 1)$. We call a polar pair of this class a
blocking pair of polyhedra, because the geometric theory
developed here has intimate connections with the notion
of a blocking pair of "clutters" defined on a finite set
E [6, 9, 10]. (A clutter on E is a family of noncomparable
subsets of E.) From this point of view, the present paper
may be regarded as a continuation of [6, 9, 10]. In
particular, it is found that the "length-width" inequality
and the "max-flow min-cut" equality, studied in [10] for a
blocking pair of clutters, are always valid for a blocking
pair of polyhedra (Theorem 3.1).

I should like to thank Jon Folkman for helpful dis-
cussion concerning the proof of Theorem 2.1.

2. A POLARITY

Let A be an m by n nonnegative matrix, and consider the convex polyhedron

$$(2.1) \qquad \mathcal{B} = \{b \in R_+^n | Ab \geq 1\}.$$

Here 1 denotes the m-vector all of whose components are 1 and R_+^n is the nonnegative orthant of R^n. The polyhedron $\mathcal{B}$ is the vector sum of the convex hull of its extreme points and the nonnegative orthant:

$$(2.2) \qquad \mathcal{B} = \text{conv. hull} (\{b^1, b^2, \ldots, b^r\}) + R_+^n,$$

where $b^1, \ldots, b^r$ are the extreme points of $\mathcal{B}$.

We say that a row vector a^i of the matrix A is in-essential if it dominates a convex combination of other rows of A, i.e., the inequality $a^i \geq \Sigma_{j=1}^m \alpha_j a^j$ holds for some $\alpha_1 \geq 0, \ldots, \alpha_m \geq 0$ satisfying $\alpha_i = 0$, $\Sigma_{j=1}^m \alpha_j = 1$; otherwise the row a^i is essential. It is a consequence of the Farkas lemma that an inequality of (2.1) may be dropped in the definition of $\mathcal{B}$ if and only if the corresponding row of A is inessential. Accordingly we may suppose without loss of generality that all rows of A are essential. We call such an A proper, and include in this definition the degenerate cases (i) A is a one-rowed zero matrix ($\mathcal{B}$ is empty), and (ii) A has no rows ($\mathcal{B} = R_+^n$).

Let

(2.3) $\hat{\mathcal{B}} = \{a \in R_+^n | a \cdot \mathcal{B} \geq 1\}$.

We call $\hat{\mathcal{B}}$ the <u>blocker</u> of $\mathcal{B}$. Note that if $\mathcal{B}$ is empty, then $\hat{\mathcal{B}} = R_+^n$, and if $\mathcal{B} = R_n^+$, then $\hat{\mathcal{B}}$ is empty. Theorem 2.1 below shows that the blocking relation is a polarity on the class of all convex polyhedra of the form (2.1).

<u>Theorem 2.1</u>. <u>Let the</u> m <u>by</u> n <u>matrix</u> A <u>be proper with</u> <u>rows</u> a^1, ..., $a^m \in R_+^n$. <u>Let</u> $\mathcal{B} = \{b \in R_+^n | Ab \geq 1\}$ <u>have extreme</u> <u>points</u> b^1, ..., b^r, <u>let B be the matrix having rows</u> b^1, ..., b^r, <u>and let</u> $\mathcal{Q} = \{a \in R_+^n | Ba \geq 1\}$. <u>Then</u> (i) $\hat{\mathcal{B}} = \mathcal{Q}$; (ii) B <u>is</u> <u>proper and the extreme points of</u> $\mathcal{Q}$ <u>are</u> a^1, ..., a^m; (iii) $\hat{\mathcal{Q}} = \mathcal{B}$.

Theorem 2.1 can be deduced from standard results about polar cones. We give a direct proof below.

<u>Proof</u>. We first prove (i). Suppose $a \in \mathcal{Q}$. Thus $b^1 \cdot a \geq 1$, ..., $b^r \cdot a \geq 1$. If $b \in \mathcal{B}$, then by (2.2) we have $b = \Sigma_{i=1}^r \alpha_i b^i + z$, where $z \geq 0$, $\alpha_i \geq 0$, $\Sigma_{i=1}^r \alpha_i = 1$. Thus $a \cdot b \geq \Sigma_{i=1}^r \alpha_i (a \cdot b^i) \geq 1$. Hence $a \in \hat{\mathcal{B}}$, and $\mathcal{Q} \subseteq \hat{\mathcal{B}}$. Conversely, if $a \in \hat{\mathcal{B}}$, then $a \cdot b \geq 1$ for all $b \in \mathcal{B}$, and in particular, $a \cdot b^1 \geq 1$, ..., $a \cdot b^r \geq 1$. Thus $a \in \mathcal{Q}$, and $\hat{\mathcal{B}} \subseteq \mathcal{Q}$. Hence $\mathcal{Q} = \hat{\mathcal{B}}$.

To show that B is proper, suppose $b^1 \geq \Sigma_{i=2}^r \alpha_i b^i$, where $\alpha_i \geq 0$, $\Sigma_{i=2}^r \alpha_i = 1$. Let $y = \Sigma_{i=2}^r \alpha_i b^i$. Then $b^1 = y + z$, $z \geq 0$. If $z = 0$, then clearly b^1 is not extreme. If $z \neq 0$,

then $y + 1/2\, z$, $y + 3/2\, z$ are distinct points of $\mathcal{B}$ whose average is b^1, again contradicting the fact that b^1 is an extreme point of $\mathcal{B}$. Hence B is proper.

Let $\mathcal{C} = $ conv. hull $(\{a^1, \ldots, a^m\})$. We shall show that $\mathcal{C} + R_+^n = \mathcal{Q}$. To this end, suppose first that $x \in \mathcal{C} + R_+^n$, so that $x = \Sigma_{i=1}^m \alpha_i a^i + z$, $z \geq 0$, $\alpha_i \geq 0$, $\Sigma_{i=1}^m \alpha_i = 1$. Then $b^j \cdot x = b^j \cdot \Sigma_{i=1}^m \alpha_i a^i + b^j \cdot z \geq 1$. Thus $x \in \mathcal{Q}$, and $\mathcal{C} + R_+^n \subseteq \mathcal{Q}$. If equality does not hold here, then by the separating hyperplane theorem, there is a $b \in R^n$ and an $\alpha \in R$ such that $b \cdot x \geq \alpha$ for all $x \in \mathcal{C} + R_+^n$, whereas $b \cdot a < \alpha$ for some $a \in \mathcal{Q}$. Since $b \cdot x \geq \alpha$ for all $x \in \mathcal{C} + R_+^n$, we must have $b \geq 0$. Thus $\alpha > b \cdot a \geq 0$. Hence $1/\alpha\, b \cdot x \geq 1$ for all $x \in \mathcal{C} + R_+^n$, and in particular, $1/\alpha\, b \cdot a^i \geq 1$, $i = 1, 2, \ldots, m$. Thus $1/\alpha\, b \in \mathcal{B}$ and hence $1/\alpha\, b = \Sigma_{i=1}^r \beta_i b^i + z$, where $z \geq 0$, $\beta_i \geq 0$, $\Sigma_{i=1}^r \beta_i = 1$. But then

$$1 > \frac{1}{\alpha}\, b \cdot a = \sum_{i=1}^r \beta_i b^i \cdot a + z \cdot a \geq 1,$$

a contradiction. Hence $\mathcal{C} + R_+^n = \mathcal{Q}$. It then follows that the row vectors $a^1, \ldots, a^m$ of A are the extreme points of $\mathcal{Q}$. For suppose a^1, say, is not extreme. Then $a^1 = 1/2(x + y)$, where $x = \Sigma_{i=1}^m \alpha_i a^i + u + a^1$, $y = \Sigma_{i=1}^m \beta_i a^i + v + a^1$, $\alpha_i \geq 0$, $\beta_i \geq 0$, $\Sigma_{i=1}^m \alpha_i = 1$, $\Sigma_{i=1}^m \beta_i = 1$, and $u \geq 0$, $v \geq 0$. Moreover, $\alpha_1 + \beta_1 < 2$, since $x \neq a^1$, $y \neq a^1$. We have

$$a^1 = \sum_{i=1}^m \frac{(\alpha_i + \beta_i)}{2}\, a^i + 1/2(u + v).$$

Let $1/2 (\alpha_1 + \beta_1) = \gamma_1$. Then $\gamma_1 < 1$ and

$$a^1 \geq \sum_{i=2}^{m} \frac{\gamma_i}{1-\gamma_1} a^i, \quad \frac{\gamma_i}{1-\gamma_1} \geq 0, \quad \sum_{i=2}^{m} \frac{\gamma_i}{1-\gamma_1} = 1.$$

This contradicts the assumption that A is proper, and finishes the proof of (ii).

Part (iii) of Theorem 2.1 now follows from (i) and (ii).

We call the matrix B of Theorem 2.1 the <u>blocking</u> matrix of A. The blocking matrix of B is then A.

An example illustrating Theorem 2.1 in R^2 is shown in Fig. 1 below.

It follows from Theorem 2.1 that if we are given the matrix A that defines $\mathcal{B}$, then the blocking matrix B defining $\mathcal{Q}$ can be determined by the following straightforward but exceedingly tedious process. Append the n by n identity matrix to A, and then find an n by n nonsingular submatrix $\overline{A}$ of the matrix thus obtained. Next solve the linear system of equations having $\overline{A}$ as coefficient matrix and having right hand side 1 or 0 according as the corresponding row of $\overline{A}$ belongs to A or to I. If the resulting solution b satisfies $b \geq 0$, $Ab \geq 1$, then b is a row of B. All rows of B are obtainable in this way.

The case in which A is a (0,1)-matrix is of particular interest for extremal combinatorics. The assumption that A is proper is then equivalent to saying that A is the m by n incidence matrix of a clutter of m subsets of a set of n elements (no row of A contains another row of A). Thus
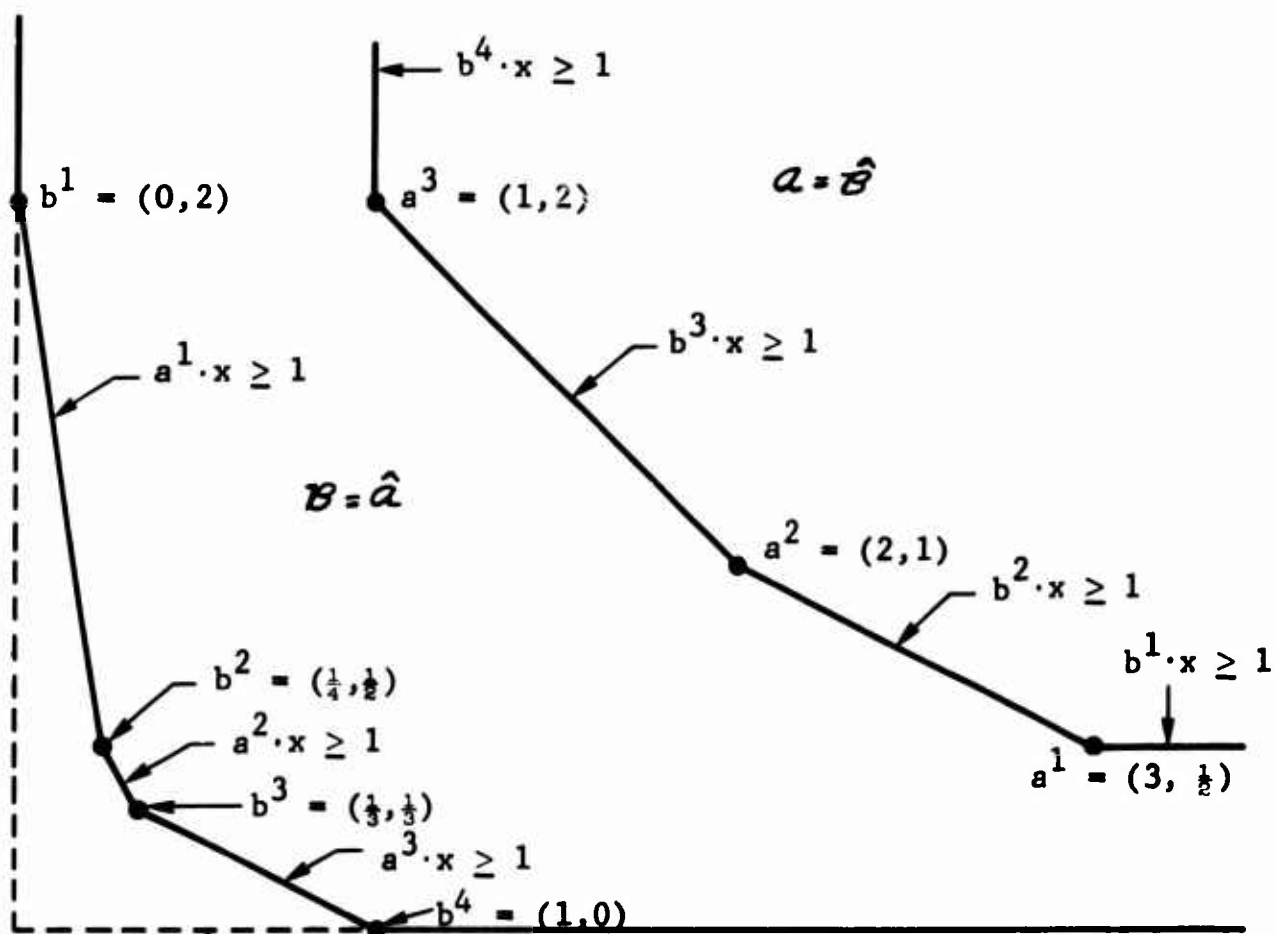
Fig. 1

Theorem 2.1 provides a way of characterizing the subsets
comprising an arbitrary clutter as the extreme points of
a convex polyhedron. If A is the incidence matrix of a
clutter, then the incidence matrix b(A) of the blocking
clutter has as rows all (0,1)-vectors with n components
that have inner product at least 1 with all rows of A and
that are minimal with respect to this property [10]. It is
not hard to see that each row of the matrix b(A) will then
be a row of matrix B in Theorem 2.1. In general, B will
have many other rows. But there are significant classes
of clutters for which b(A) = B. For example, if A is the
incidence matrix of all simple paths joining two distinguished
nodes of a graph having n edges, then b(A) is equal to B,
and hence B is the incidence matrix of all cuts separating
the two nodes. (By the duality asserted in Theorem 2.1, we
could of course start with b(A) and obtain A.)

In the context of Theorem 2.1, Lehman's interesting
paper [10] can be viewed as a study of clutters A for which
b(A) = B. Generalizing from the example above, he shows
that this situation holds for a clutter A if and only if
the max-flow min-cut equality holds for A and b(A), or if
and only if the length-width inequality holds for A and b(A).
Here, in analogy with the example of paths and cuts in a
graph, the max-flow min-cut equality is said to hold for
a (0,1)-matrix A corresponding to a clutter, and the
matrix b(A) having rows $b^1, \ldots, b^r$, if and only if for

every $w \in R_+^n$, it is true that in the linear program

(2.4) $yA \leq w,$

  $y \geq 0,$

  $\max 1 \cdot y,$

we have

(2.5) $\max 1 \cdot y = \min_{1 \leq i \leq r} b^i \cdot w.$

Similarly, the length—width inequality is said to hold for a (0, 1)—matrix A, whose rows $a^1, \ldots, , a^m$ correspond to the sets of a clutter, and the matrix b(A), if and only if for every $\iota \in R_+^n$, $w \in R_+^n$, we have

(2.6) $\left(\min_{1 \leq i \leq m} a^i \cdot \iota \right) \left(\min_{1 \leq i \leq r} b^i \cdot w \right) \leq \iota \cdot w.$

In the next section we shall examine the analogs of (2.5) and (2.6) for a nonnegative matrix A and the blocking matrix B of Theorem 2.1.

3. THE LENGTH-WIDTH INEQUALITY AND MAX-FLOW MIN-CUT EQUALITY.

Let A and B be m by n and r by n proper matrices having rows $a^1, \ldots, a^m$ and $b^1, \ldots, b^r$ respectively. Say that the max-flow min-cut equality holds for the pair A, B (in this order) if and only if, for each $w \in R_+^n$, it is true that in the linear program

(3.1)
$$yA \leq w,$$
$$y \geq 0,$$
$$\max 1 \cdot y,$$

we have

(3.2)
$$\max 1 \cdot y = \min_{1 \leq i \leq r} b^i \cdot w.$$

Similarly, say that the length-width inequality holds for A and B if and only if, for every $l \in R_+^n$, $w \in R_+^n$, we have

(3.3)
$$\left(\min_{1 \leq i \leq m} a^i \cdot l \right) \left(\min_{1 \leq i \leq r} b^i \cdot w \right) \leq l \cdot w.$$

Theorem 3.1. (i) Let A and B be a pair of blocking matrices. Then the max-flow min-cut equality holds for A, B (in either order) and the length-width inequality holds for A, B.

(ii) Let A and B be proper matrices whose rows $a^1, \ldots, a^m$ and $b^1, \ldots, b^r$ satisfy $a^i \cdot b^j \geq 1$. If the length-width inequality holds for A, B, then A and B are

<u>a blocking pair</u>.

(iii) <u>Let</u> A <u>and</u> B <u>be proper matrices</u>. <u>If the max-flow min-cut equality holds for</u> A, B <u>(in that order)</u>, <u>then</u> A <u>and</u> B <u>are a blocking pair</u>.

<u>Proof</u>. (i) Suppose that A and B are a blocking pair. The max-flow min-cut equality follows from Theorem 2.1 and the duality theorem for linear programs. The linear program dual to (3.1) is

$$(3.4) \qquad \begin{aligned} Ab &\geq 1, \\ b &\geq 0, \\ \min\ w\cdot b. \end{aligned}$$

The minimum in this program is achieved at an extreme point of the constraint set $\mathcal{B} = \{b \in R_+^n | Ab \geq 1\}$, that is, by Theorem 2.1, at a row of B, and hence (3.2) holds.

To see that the length-width inequality holds, let

$$(3.5) \qquad \lambda = \min_{1 \leq i \leq m} a^i \cdot \ell = \min_{a \in \mathcal{a}} a \cdot \ell,$$

$$(3.6) \qquad \omega = \min_{1 \leq j \leq m} b^j \cdot w = \min_{b \in \mathcal{B}} b \cdot w.$$

Here $a^1, \ldots, a^m$ are the rows of A, $b^1, \ldots, b^r$ are the rows of B, and $\mathcal{a} = \{a \in R_+^n | Ba \geq 1\}$, $\mathcal{B} = \{b \in R_+^n | Ab \geq 1\}$. The second equality in (3.5) (the second equality in (3.6)) follows from Theorem 2.1 and the fact that the minimum

value of a nonnegative linear form defined over $\mathcal{a}(\mathcal{B})$ occurs at an extreme point of $\mathcal{a}(\mathcal{B})$.

If either $\lambda = 0$ or $\omega = 0$, the length-width inequality holds trivially. If both $\lambda \neq 0$, $\omega \neq 0$, then we have $\ell/\lambda \cdot a \geq 1$ for all $a \in \mathcal{a}$ and $w/\omega \cdot b \geq 1$ for all $b \in \mathcal{B}$ by (3.5) and (3.6). Consequently $\ell/\lambda \in \hat{\mathcal{a}}$ and $w/\omega \in \hat{\mathcal{B}} = \mathcal{a}$. Thus $\ell/\lambda \cdot w/\omega \geq 1$, $\ell \cdot w \geq \lambda\omega$.

(ii) Let A and B be proper matrices whose rows satisfy $a^i \cdot b^j \geq 1$, and define $\mathcal{B} = \{x \in R_+^n | Ax \geq 1\}$, $\mathcal{a} = \{x \in R_+^n | Bx \geq 1\}$. Thus $\hat{\mathcal{B}} = \text{conv. hull } (\{a^1, \ldots, a^m\}) + R_+^n$ and $\hat{\mathcal{a}} = \text{conv. hull } (\{b^1, \ldots, b^r\}) + R_+^n$ satisfy $\hat{\mathcal{a}} \cdot \hat{\mathcal{B}} \geq 1$. Hence $\hat{\mathcal{a}} \subseteq \hat{\hat{\mathcal{B}}} = \mathcal{B}$. Assume now that the length-width inequality holds for A, B, and let $b \in \mathcal{B}$. We want to show that $b \cdot \mathcal{a} \geq 1$. Thus let $a \in \mathcal{a}$. By the length-width inequality applied to a, b, we have

$$a \cdot b \geq \left(\min_{1 \leq j \leq r} a \cdot b^j\right)\left(\min_{1 \leq i \leq m} b \cdot a^i\right) \geq 1,$$

since $a \in \mathcal{a}$, $b \in \mathcal{B}$. Thus $\mathcal{B} \subseteq \hat{\mathcal{a}}$, and hence $\hat{\mathcal{a}} = \mathcal{B}$.

(iii) Let A and B be proper matrices and assume that the max-flow min-cut equality holds for A, B. Let $a^1, \ldots, a^m$ be the rows of A, let $b^1, \ldots, b^r$ be the rows of B, and let $\mathcal{B} = \{x \in R_+^n | Ax \geq 1\}$. Suppose that $b^j \notin \mathcal{B}$. By the separating hyperplane theorem, there is a $w \in R_+^n$ and an $\alpha > 0$ such that $w \cdot b^j < \alpha \leq w \cdot b$ for all $b \in \mathcal{B}$. But by the duality theorem for linear programs and the max-flow min-cut

equality, we have $\min\limits_{b \in \mathscr{B}} b \cdot w = \min\limits_{1 \leq i \leq r} b^i \cdot w$, a contradiction. Hence $b^j \in \mathscr{B}$ for $1 \leq j \leq r$ and consequently $a^i \cdot b^j \geq 1$ for $1 \leq i \leq m$, $1 \leq j \leq r$. We shall finish the proof by showing that the length–width inequality holds. Thus let $\ell \in R_+^n$, $w \in R_+^n$, and define

$$\lambda = \min_{1 \leq i \leq m} a^i \cdot \ell, \quad \omega = \min_{1 \leq j \leq r} b^j \cdot w.$$

By the max–flow min–cut equality, there is a $y = (\eta_1, \ldots, \eta_m) \geq 0$ such that $yA \leq w$ and $1 \cdot y = w$. Thus

$$\lambda\omega = \lambda(1 \cdot y) = \lambda \sum_{i=1}^{m} \eta_i \leq \sum_{i=1}^{m} (a^i \cdot \ell)\eta_i = \ell \cdot \sum_{i=1}^{m} \eta_i a^i \leq \ell \cdot w.$$

Hence the length–width inequality holds for A, B, and thus A, B are a blocking pair.

Theorem 3.1 is sometimes useful in proving that two matrices A and B are a blocking pair. In Sec. 6, for example, we shall take A to be the incidence matrix of the clutter of permutation matrices and use Theorem 3.1 (iii) to pin down the blocking matrix B. Some other examples of this kind will also be discussed.

4. CONTRACTIONS, DELETIONS, PAINTINGS.

One can define operations of "contracting a coordinate" or "deleting a coordinate" in a proper matrix A (or on the polyhedron $\mathcal{B} = \{b \in R_+^n | Ab \geq 1\}$) that are analogous to the operations of "contracting an element" or "deleting an element" in a graph, a matroid [13], or a clutter [10]. Just as for graphs, matroids, or clutters, these operations commute. Moreover, contracting the i-th coordinate in A corresponds to deleting the i-th coordinate in its blocking matrix B: the resulting matrices again constitute a blocking pair.

Let A be an m by n proper matrix. By a <u>contraction of coordinate</u> $i \in \{1, \ldots, n\}$ <u>in</u> A, we mean the following: drop the i-th column of A, and then drop all inessential rows in the resulting matrix. A <u>deletion of coordinate</u> i <u>in</u> A is the following: drop the i-th column of A, and then drop all rows that had a positive entry in column i. The new matrix obtained in each case is proper.

Geometrically, contracting coordinate i in A is an intersection of the polyhedron $\mathcal{B} = \{b \in R_+^n | Ab \geq 1\}$ with the hyperplane $\varsigma_i = 0$; deleting coordinate i is a projection of $\mathcal{B}$ on the hyperplane $\varsigma_i = 0$. It is easy to see that first contracting coordinate i, then deleting coordinate j, is equivalent to first deleting coordinate j, then contracting coordinate i. Thus one can unambiguously define "minors" of A (or of $\mathcal{B}$), just as in matroid theory [13], that arise by contracting some subset of coordinates and deleting some

other subset, since the order in which operations are carried out is immaterial.

It is also not hard to see that if A, B are a blocking pair, and if coordinate i is contracted in A, then the blocker of the resulting matrix is obtained from B by deleting coordinate i.

If A and b(A) are n—columned incidence matrices of a blocking pair of clutters, and if we partition the set $\{1, 2, \ldots, n\}$ into two sets, the "blue" set and the "red" set, say, then it is true that precisely one of the following alternatives holds: (a) there is a row of A all of whose 1's lie in the blue set; (b) there is a row of b(A) all of whose 1's lie in the red set. This "painting theorem" in fact characterizes the blocking relation for a pair of clutters on $\{1, 2, \ldots, n\}$ [6, 9]. The analogous painting theorem is valid also for blocking matrices A and B: For any partition of the column set $\{1, 2, \ldots, n\}$ into two sets, blue and red (empty sets not being excluded), there is either a row vector of A whose support is blue or a row vector of B whose support is red, but not both. (Here the support of a vector $a = (\alpha_1, \ldots, \alpha_m)$ is the set of i ϵ $\{1, 2, \ldots, n\}$ such that $\alpha_i \neq 0$.) It is clear that both alternatives cannot hold, since otherwise some row of A and some row of B would have inner product zero. That one of the alternatives must hold can be seen in various ways, e.g., assume there is no row of A whose support is blue, and consider the effect of deleting all red coordinates in A. Or consider the max—flow min—cut

equality for A, B where the vector w has red coordinates
zero and blue coordinates one.

In terms of the blocking polyhedra $\mathcal{B} = \{b \in R_+^n | Ab \geq 1\}$
and $\mathcal{A} = \{a \in R_+^n | Ba \geq 1\}$ defined by blocking matrices A and B,
the painting theorem asserts that for any blue—red partition
of the coordinates, there is either an $a \in \mathcal{A}$ with blue support
or a $b \in \mathcal{B}$ with red support, but not both.

5. BLOCKING POLYHEDRA FROM ORTHOGONAL COMPLEMENTS.

A particular class of blocking polyhedra can be generated in the following way. Let $\mathcal{R}$ and $\mathcal{R}*$ be complementary orthogonal subspaces of R^n, and let $\mathcal{E}$ and $\mathcal{E}*$ be the set of all elementary vectors of $\mathcal{R}$ and $\mathcal{R}*$, respectively. (Here a vector of space $\mathcal{R}$ is elementary if it is nonzero and has minimal support over all nonzero vectors of $\mathcal{R}$[13]). Define

$$(5.1) \qquad \mathcal{E}_1 = \{a = (\alpha_1, \ldots, \alpha_n) \in \mathcal{E} | \alpha_1 = 1\},$$

$$(5.2) \qquad \mathcal{E}_1* = \{b = (\beta_1, \ldots, \beta_n) \in \mathcal{E}* | \beta_1 = 1\}.$$

The sets $\mathcal{E}_1$ and $\mathcal{E}_1^*$ are finite, say

$$(5.3) \qquad \mathcal{E}_1 = \{a^1, a^2, \ldots, a^m\},$$

$$(5.3) \qquad \mathcal{E}_1^* = \{b^1, b^2, \ldots, b^r\}.$$

For each $a^i = (1, \alpha_2, \ldots, \alpha_n) \in \mathcal{E}_1$, $b^j = (1, \beta_2, \ldots, \beta_n) \in \mathcal{E}_1^*$, let

$$(5.5) \qquad \bar{a}^i = (|\alpha_2|, \ldots, |\alpha_n|),$$

$$(5.6) \qquad \bar{b}^j = (|\beta_2|, \ldots, |\beta_n|),$$

and let A and B be the nonnegative matrices with $n - 1$ columns having rows $\bar{a}^1, \ldots, \bar{a}^m$ and $\bar{b}^1, \ldots, \bar{b}^r$, respectively. It is

easy to check that A and B are proper. We assert that A and B constitute a blocking pair of matrices.

This does not seem to be obvious, although one can see quickly that $\bar{a}^i \cdot \bar{b}^j \geq 1$, since $a^i \cdot b^j = 0$. That matrices A and B are a blocking pair follows from Theorem 3.1 (iii) and the proof in [9] that the max-flow min-cut equality holds for A, B.

One case of particular interest for combinatorics is where the space $\mathcal{R}$ (and hence $\mathcal{R}*$) is regular, i.e., can be viewed as the row space of a totally unimodular matrix [13]. (A matrix is totally unimodular if all its square submatrices have determinant 0 or ± 1.) In this case each elementary vector of $\mathcal{R}$ (and of $\mathcal{R}*$) can be taken to have coordinates 0, 1, or -1, and consequently matrices A and B above are (0, 1)-matrices. For example, if the space $\mathcal{R}$ is the row space of the (0, ± 1) vertex-edge incidence matrix of an oriented graph on n edges, the construction above yields A as the (0,1)-incidence matrix of all cuts separating the two end nodes of edge 1 in the underlying unoriented graph with edge 1 suppressed, and B as the incidence matrix of all paths joining these two nodes in the same graph.

6. OTHER EXAMPLES OF BLOCKING POLYHEDRA.

In this concluding section we describe some other examples of blocking polyhedra that have combinatorial interest. In each of the examples, we start with a (0,1)-matrix A which can be viewed as the incidence matrix of a clutter on a finite set, and examine the blocking matrix B. It is usually difficult to determine B, and we have not succeeded in doing this for certain clutters about which little is known, such as all minimal node-covers in an arbitrary graph, or all Hamiltonian tours in a complete graph.

Let A be the incidence matrix of all n by n permutation matrices. Thus A has $n!$ rows and n^2 columns corresponding to pairs i, j, for i, $j \in \{1, 2, \ldots, n\}$. We assert that the blocking matrix B consists of the essential rows of the following matrix B^+. For each $I \subseteq \{1, 2, \ldots, n\}$, $J \subseteq \{1, 2, \ldots, n\}$ such that $s(I, J) = |I| + |J| - n > 0$, let $b(I, J)$ be the n^2-vector having coordinates $1/s(I, J)$ for $i \in I$, $j \in J$, zero otherwise, and let B^+ be the matrix consisting of all rows $b(I, J)$. (Some of the rows of B^+ are inessential, but not many. If $I = \{1, 2, \ldots, n\}$, and $J = \{j_1, \ldots, j_k\}$ is not a singleton, the row is inessential, being a convex combination of the rows $b(I, \{j_1\})$, $\ldots$, $b(I, \{j_k\})$, and similarly for I not a singleton, $J = \{1, 2, \ldots, n\}$. It can be shown that all other rows of B^+ are essential, however.) That the matrix B is the blocking matrix of A follows from Theorem 3.1 (iii) and results of [8], where

the max-flow min-cut equality is proved for the matrices A and B^+, and hence for the proper matrices A and B. As shown in [8], there is an efficient algorithm for solving the max-flow min-cut problem for the matrices A and B, in this order, based on the maximum flow routine for (ordinary) flows in networks. That is, the maximizing vector y in (3.1) and the minimizing row of B in (3.2) can be calculated explicitly and efficiently. The max-flow min-cut equality of course holds in the reverse order B, A, but we know of no efficient algorithm for finding the maximizing vector y here. Finding the minimizing row of A is the well-known optimal assignment problem, for which efficient methods are known. It seems likely that there is an alternative approach to the optimal assignment problem based on the max-flow min-cut equality for B, A, i.e., based on the above characterization of the permutation matrices as the extreme points of $a = \{a \in R_+^{n^2} \mid Ba \geq 1\}$. For example, consider the 7 by 7 assignment problem with cost matrix w shown in Fig. 2 below. An optimal assignment (minimizing row of A) is indicated by the asterisks in the figure. An optimal y weights two rows of B positively: $y(\{6,7\}, \{2,3,4,5,6,7\}) = 1$, $y(\{2,4,5,6,7\}, \{3,5,6,7\}) = 2$.

The discussion above can be generalized to the linear programming problem known as the transportation problem [7].

	1	2	3	4	5	6	7
1	3	6	2	0	0	0	0*
2	8	0*	4	0	2	3	6
3	9	2	0*	8	5	3	0
4	2	0	4	4	1*	5	5
5	0	9	7	3	8	1*	9
6	0*	3	4	5	8	8	5
7	0	9	9	1*	7	6	3

Fig. 2

We turn next to an example of a different kind. Let
A be the incidence matrix of all minimal node—covers in a
graph. It would be interesting to know a characterization
of the row vectors of A as the extreme points of a poly—
hedron. Here, in contrast with the example above, in—
equalities characterizing the convex hull are not known.
Neither do we know inequalities characterizing the vector
sum of the convex hull and the nonnegative orthant, i.e.,
the rows of the blocking matrix B. Some examples may
indicate the difficulty of determining B. Suppose A is
the incidence matrix of all minimal covers in the complete
graph on n vertices. Thus A is n by n with zeros down the
main diagonal and ones in all other positions. It is not
difficult to see in this case that the matrix B has $2^n - (n+1)$
rows, one corresponding to each subset $I \subseteq \{1, 2, \ldots, n\}$

such that $|I| \geq 2$; specifically, the row b(I) has coordinates $1/(|I|-1)$ in positions corresponding to $i \in I$, zeros elsewhere. Thus here again we have the situation in which the nonzero elements of each row of B are all equal, just as for the permutations. But this situation is not typical for this problem. For instance, consider the graph of a "wheel" with an odd number of "spokes." To be specific, consider a 5-sided wheel, shown in Fig. 3 below. The matrices A and B are shown in Fig. 3 also. Note the first row of B.

A

1	2	3	4	5	6
0	1	1	1	1	1
1	1	1	0	1	0
1	0	1	1	0	1
1	1	0	1	1	0
1	0	1	0	1	1
1	1	0	1	0	1

B

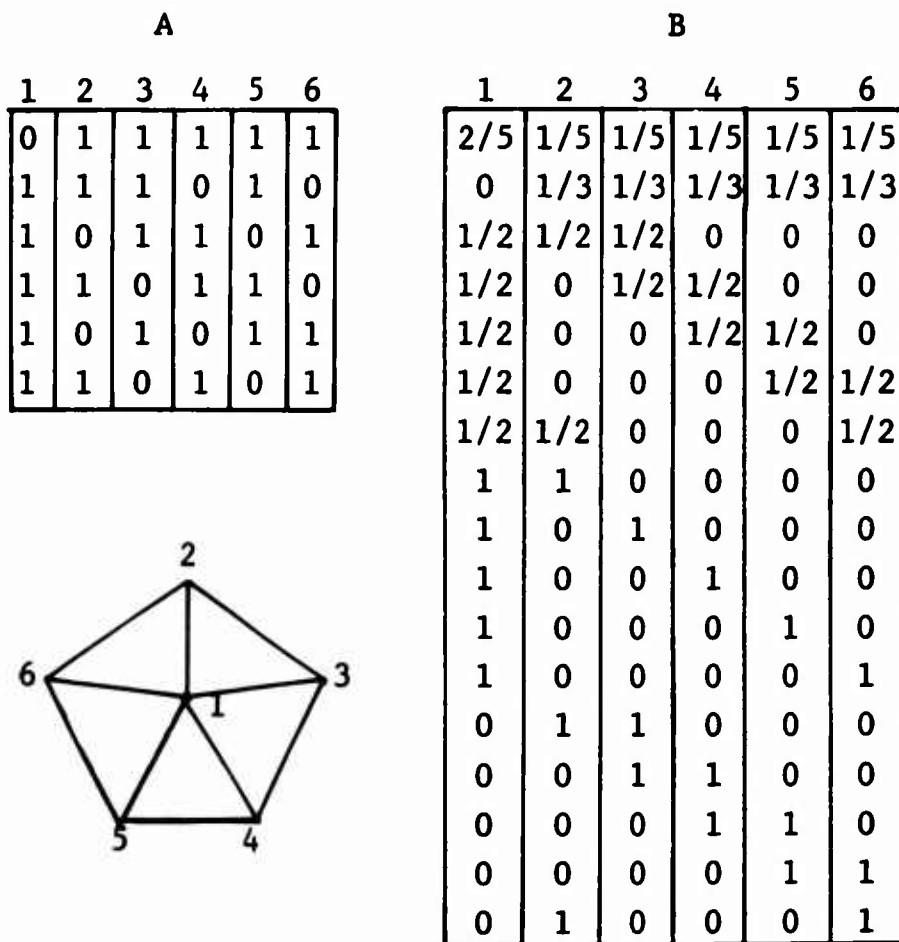| 1	2	3	4	5	6
2/5	1/5	1/5	1/5	1/5	1/5
0	1/3	1/3	1/3	1/3	1/3
1/2	1/2	1/2	0	0	0
1/2	0	1/2	1/2	0	0
1/2	0	0	1/2	1/2	0
1/2	0	0	0	1/2	1/2
1/2	1/2	0	0	0	1/2
1	1	0	0	0	0
1	0	1	0	0	0
1	0	0	1	0	0
1	0	0	0	1	0
1	0	0	0	0	1
0	1	1	0	0	0
0	0	1	1	0	0
0	0	0	1	1	0
0	0	0	0	1	1
0	1	0	0	0	1

Fig. 3

Another clutter arising in graph theory that has been studied in depth by Edmonds [1, 2] is where A is the incidence matrix of all perfect matchings in a complete graph. Edmonds has characterized the convex hull of such a clutter, and has described an efficient algorithm for determining a minimum-weight perfect matching for an arbitrary weight function defined on the edges of the graph. What is the blocking matrix B for A? The blocking clutter b(A) is described in [6], the description being deduced from Tutte's theorem characterizing graphs that contain a perfect matching [11]. All members of b(A) will yield rows of B, but what other kinds of rows does B have? It would appear that to answer this question, we need information about the maximum "number of disjoint matchings" contained in an arbitrary graph, at least in the sense of admitting rational weights on matchings, i.e., we need to know how to solve the max-flow problem (3.1) for A and an arbitrary $w \geq 0$ defined on the edges of the graph. Good information on the (0, 1)-form of this problem (i.e., w a given (0,1)-vector and y restricted to be a (0,1)-vector) could well lead to a solution of the four-color problem (the three-color problem for edges of a cubic planar graph). Even the rational form of the problem appears to be unsolved, except in the bipartite case, where we are dealing with the permutation matrices.

In each of the examples discussed thus far in this section, the blocking matrix B for the incidence matrix A of a clutter appears to be more complicated in structure than a list of inequalities defining the convex hull of the clutter. We conclude with two examples in which this is not the case.

Let A be the incidence matrix of all spanning trees in a graph on edge-set $\{1, 2, \ldots, n\}$ (more generally, we could consider bases in a matroid). Here Edmonds has shown that the extreme points of the polyhedron

$$(6.1) \qquad \sum_{i=1}^{n} \zeta_i = \text{rank} (\{1, 2, \ldots, n\}),$$

$$(6.2) \qquad \sum_{i \in I} \zeta_i \leq \text{rank} (I), \quad \text{all } I \subseteq \{1, 2, \ldots, n\},$$

$$(6.3) \qquad \zeta_i \geq 0,$$

are precisely the rows of A [5]. (It is enough to consider sets $I \subseteq \{1, \ldots, n\}$ in (6.2) that are spans.) It can also be shown, using results of Tutte [12] and of Edmonds [3, 4], that the blocking matrix B of A consists of the essential rows of the matrix B^+ which has a row $b(\bar{I})$ corresponding to each nonempty complement $\bar{I}$ of a span I: the row $b(\bar{I})$ has components $1/(\text{rank} (\{1, 2, \ldots, n\}) - \text{rank} (I))$ in positions corresponding to elements of $\bar{I}$, zeros elsewhere. Edmonds has described an efficient algorithm for finding the

maximum number of edge-disjoint spanning trees in a graph
[3, 4] (or the maximum number of disjoint matroid bases) and
it is not hard to extend this to the case of an arbitrary
weight function $w \geq 0$ defined on the edges. In other words,
the max-flow problem (3.1) in either integer or rational form
has been solved for this clutter A, just as it has for the
clutter of permutation matrices. The max-flow problem in
the other direction, that is, for B, A, has also been solved:
Finding a min-cut is the well-known minimum spanning tree
problem and it is not difficult to describe an algorithm
for finding a corresponding maximizing vector y. Here one
can make good use of contractions and deletions in determin-
ing a min-cut and a max-flow.

Our last example deals with edge-covers in a bipartite
graph. Let A be the incidence matrix of all minimal covers
of nodes by edges in a bipartite graph having r nodes in
one part, s in the other, and n edges. (Since we want to
consider an arbitrary weight function $w \geq 0$ defined on the
edges, we could suppose without loss of generality that the
graph is a complete bipartite graph having rs edges. Then
A has rs columns and a row corresponding to every minimal
cover, i.e., a row corresponding to every r by s (0, 1)-
matrix having at least one 1 in each of its rows and columns,
and which is minimal with respect to this property.) The
blocking matrix B of A here is simply the node-edge incidence
matrix of the graph. In other words, the incidence matrix

b(A) of the blocking clutter is equal to B in this instance.
This can be shown in various ways, perhaps the easiest of
which is to start with the (0, 1)-matrix B and ask for its
blocking matrix. It is well-known that the matrix B is
totally unimodular, and it follows from this that all
extreme points of $\alpha = \{a \in R_+^n | Ba \geq 1\}$ are (0, 1)-vectors.
Consequently, the blocking matrix A of B is the one described
above. In connection with this example, the max-flow min-
cut equality for A, B (in this order) appears to be a new
result. Moreover, it can be shown that if $w \geq 0$ has integer
coordinates, then there is a maximizing y in (3.1) having
integer coordinates, which leads to the following theorem:
The maximum number of edge-disjoint covers (of nodes by
edges) in a bipartite graph is equal to the minimum valence
in the graph. This appears to be an overlooked companion
to the well-known König theorem that a bipartite graph
having maximum valence k can be decomposed into a sum of
k matchings, i.e., the minimum number of colors required in
an edge-coloring is equal to the maximum valence. In
terms of (0, 1)-matrices, the König theorem says that if
G is a given (0, 1)-matrix, then the least k for which we
have

(6.4) $$G \leq M_1 + \ldots + M_k,$$

where each M_i is a (0, 1)-matrix having at most one 1 in

each row and column, is equal to the largest row or column
sum of G. The companion theorem says that the largest k
for which

(6.5) $G \geq C_1 + \ldots + C_k,$

where each C_i is a (0, 1)-matrix having at least one 1 in
each row and column, is equal to the smallest row or column
sum of G.

We have said nothing about the length-width inequality
in these examples. In the examples where the blocking
matrix B is known (for permutations, trees, and edge-covers
in a bipartite graph), the corresponding length-width in-
equality appears to be a new result in each case.

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A geometric theory of "blocking polyhedra" is developed and applied to a number of problems in extremal combinatorics. The basic notion is a variant of the concept of polar polyhedra and exhibits a similar duality. The max-flow min-cut equality and the length-width inequality, valid for paths and cuts in a network, always hold for a blocking pair of polyhedra, and the former of these characterizes the blocking relation. A typical combinatorial application is a new geometric characterization of the permutation matrices as the extreme points of a certain unbounded convex polyhedron.	Mathematics Probability Combinatorics Network theory